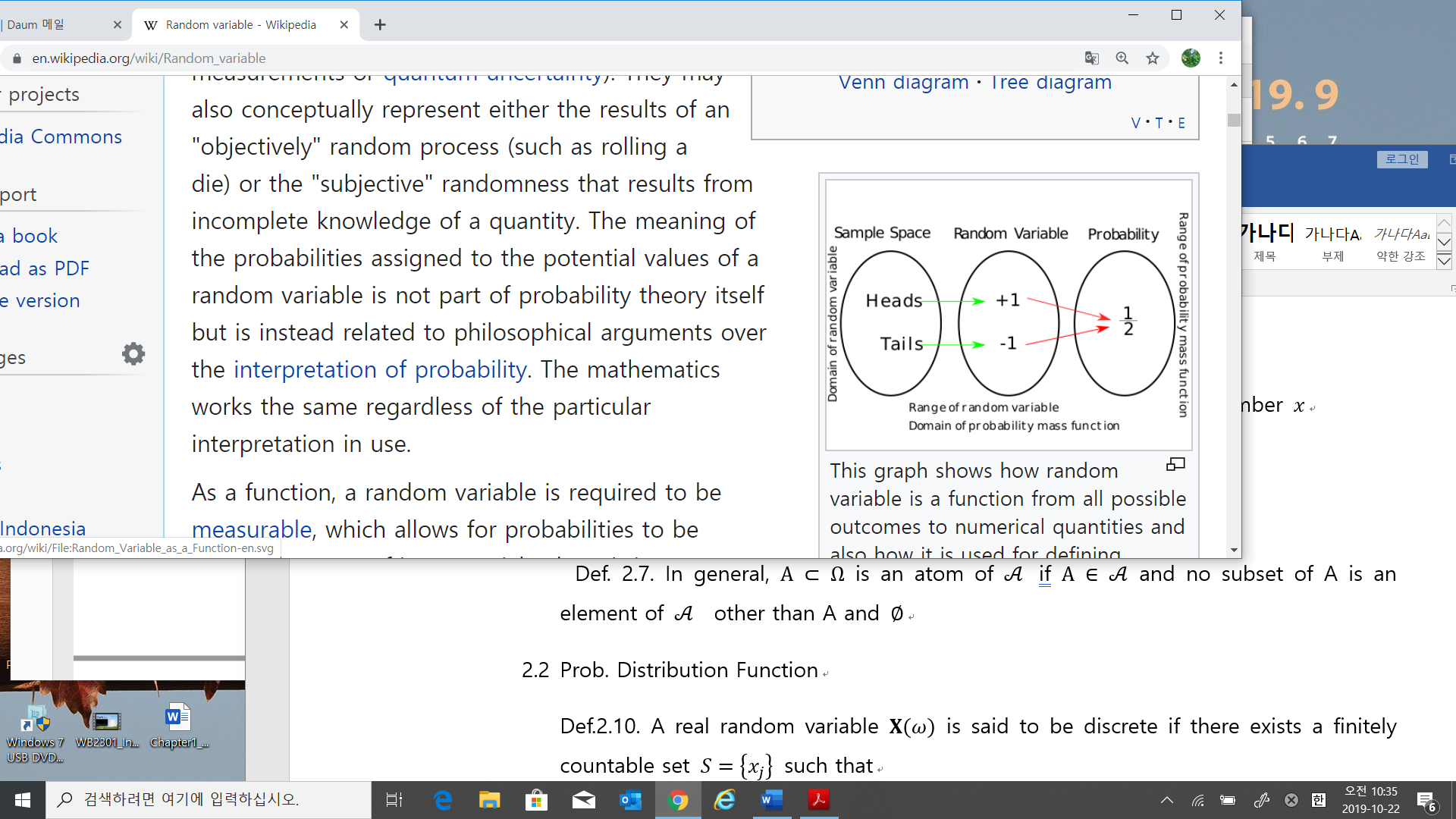
2. Random Variables and Stochastic Process

2.1 Random Variables

Def. 2.1. Given a probability space,, a random variable is a real-(vector-) valued point function which carries a sample point, , into a point in such a way that every sets, , of the form

is an element of the

* The textbook, misspelled. As
* Random variable is a function such that associates to a real number
* Wiki



* Random variables   
   -sample space

-sample point

-event: pick up one student 🡪 .

- probability =

1) ,

2) ,

Def. 2.7. In general, is an atom of if and no subset of A is an element of other than A and

* See the notation ,
* Skip from Ex.2. to the last of 2.1
  1. Prob. Distribution Function(Probability Cumulative Function, CDF)
* Probability Distribution function CDF
* Joint probability distribution function( joint cumulative distribution function)

Let . Then the Joint CDF is

1. The properties of CDF

* is a non-decreasing function, that is,
* is a right-continuous function, s.t.,
* The proofs are in

<http://www.maths.qmul.ac.uk/~bb/MS_Lectures_3and4.pdf>

* The probability of the decomposition of sets

Let so that

Then, using Axiom 2, (1.4), we get

)

)

Hence

* The discontinuity of

If is discontinuous at ,

Ex. 2.4 and Ex 2.9.

Def.2.10. A real random variable is said to be discrete if there exists a finitely countable set such that

* 1. Prob. Density Function
* Suppose , integrable function.

Then is the probability density function(PDF).

Hence is also

Proposition 2.12

* Absolute continuous / continuous 🡪 skip
* Common Distributions Functions for Random Variables

1. Uniform distribution functions
2. Exponential distribution function
3. Gaussian probability distribution.

where

The CDF of Gaussian has not closed form,

For a random vector , the PDF is

where is the covariance matrix, is the determinant of

* 1. Probabilistic Concepts Applied to Random Variables

Def 2.16. Two random variables and are called **independent** if any event of the form is independent of any event of the form where are sets in

* The joint CDF of independent RVs

Let are independent RVs, then

* Marginal CDF : see textbook p.37
  1. Functions of a RV
* PDF of the function of RV

Assume : a continuous function such that

exists

2) and have continuous partial derivatives

Then

Prop. 2.17 Given and and a vector with density function , the n-vector

has the density function

where stands for the absolute value of the determinant of the matrix

* is called a Jacobian
  1. Remind of the change variable method (Jacobian as a scaler)

Define a new variable, and substitute into

* 🡪 is a “Jacobian”
  1. Look at (2.17.a)

Given , find 🡪

Remember

Ex. 2.18

Given independent two RVs . Let find

* Solution

Define to get

Thus,

The determinant of Jacobian is

Hence

Using the marginal probability property

* Please remember (Ex.2.18.a)
  1. Expectations and Moments of a Random Variable
* Expectation

1. Mean (or the first moment or expected value) of RV
2. Expectation operator
3. Sample mean of independent measurements

* Ex. 2.19
* Ex. 2.20 / Ex. 2.21
* The expected value is interpreted as being the center of mass
* The expectation of constant
* The expectation operator is linear
* Mean square or second moment
* Higher-order moments
* Higher moments of Gaussian RV.
* In text book comments, p43. “we almost never see “
* In correct statement : “yes” , we see 4th moments as “kurtosis” which is an indicator for

Gaussian RV. You may see in “Data Science” , “Statistics”

* Variance / standard deviation
* Sample variance
* Why not ? Answer will be in exercise.
* Covariance of and
* Hence
* **Prob: If are independent, then** ,
* Correlation
* Covariance matrix for random vectors 🡪 see text book p.59
* Markov’s Inequality ( for RV’s PDF )
* Chevbyshev inequality
* Law of large number

Let measurements of as which is independent identically distributed (i.i.d). Then the sample mean is

And

Using Chevbyshev’s inequality

1. Weak law of large numbers
2. Strong law of large numbers
   1. Characteristic Functions

* Definition: given X,
* The Fourier transform of the PDF
* Ex.2.26: convolution of sum of RVs

Proof using the characteristic functions:

Make the change of variable . Hence

* Lemma 2.27

Proof: By the definition

so that

At

* Prop. 2.28: , then

Proof: See the text book P.49

* Prop. 2.29 Uncorrelated Gaussian RVs are independent
* Thrm. 2.30 : Given .Then is ,
* Higher moments: textbook p.51
* Thm. 2.31 The central limit theorem: textbook p.51
  1. Conditional Expectations and Conditional Probabilities
* Skip form the front to Ex.2.33
* Properties of Conditional probability

1. Conditional probability
2. Conditional probability distribution
3. Related Conditional probability formula

If are independent RVs

1. Total expectation law (proof: text book p.57)

* Lemma 2.34

1. are independent RVs

* Comments

1. For a discrete RV, the conditional expectation is

* is the mean of given fixed
* is a Random Variable!! , is not determined, is a function of

1. Example of the conditional expectation

Suppose ,

Then Calculate

1. then ,

then

1. then ,

then

1. Hence

🡪 is a RV w.r.t the

* 1. Stochastic Process
* Def 2.36: A stochastic process is a family of random variables, indexed by real number and defined on a common probability space
* Notation:

+

* If the natural number, then is called a random sequence.
* If , then is called a random function or Process.
* If fixed and let vary over , i.e.,

which is a RV

* If fix and let vary over , we get a sample path or realization.
* Def. 2.38

1. A stochastic process is said to be continuous in probability at if
2. Separable 🡪 skip ..

* Def. 2.42. Let be a RP defined on Let be a partition of the If the increments, are mutually independent for any partition of ,then is said to be a process with independent increments.
* Def.2.43. is a Gaussian process if for every finite collection, , the corresponding density function,

is a Gaussian density function.

* Def. 2.44. A RP is a Gaussian process if every finite linear combination of the form

is a Gaussian RV.

* Def. 2.45: A RP where is a subset of the real line, is a Markov process if for any increasing collection
* A property of Markov process: The Transition probability density function.
  1. Gauss-Markov Process

Given a GM process

* Assumption

1. , a gaussian random sequence vector

where

1. Initial condition ( Gaussian)
2. Initial condition is uncorrelated with noise i.e.,

* The transition probability
* The mean dynamic equation
* Define mean of as .
* The covariance dynamic equation
  1. Nonlinear Stochastic Difference Equations. –Skip